

The conformal status of $\omega = -3/2$ Brans-Dicke cosmology.

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Following recent fit of supernovae data to Brans-Dicke theory which favours the model with $\omega = -3/2$ [1] we discuss the status of this special case of Brans-Dicke cosmology in both isotropic and anisotropic framework. It emerges that the limit $\omega = -3/2$ is consistent only with the vacuum field equations and it makes such a Brans-Dicke theory conformally invariant. Then it is an example of the conformal relativity theory which allows the invariance with respect to conformal transformations of the metric. Besides, Brans-Dicke theory with $\omega = -3/2$ gives a border between a standard scalar field model and a ghost/phantom model.

In this paper we show that in $\omega = -3/2$ Brans-Dicke theory, i.e., in the conformal relativity there are no isotropic Friedmann solutions of non-zero spatial curvature except for $k = -1$ case. Further we show that this $k = -1$ case, after the conformal transformation into the Einstein frame, is just the Milne universe and, as such, it is equivalent to Minkowski spacetime. It generally means that only flat models are fully consistent with the field equations. On the other hand, it is shown explicitly that the anisotropic non-zero spatial curvature models of Kantowski-Sachs type are admissible in $\omega = -3/2$ Brans-Dicke theory. It then seems that an additional scale factor which appears in anisotropic models gives an extra degree of freedom and makes it less restrictive than in an isotropic Friedmann case.

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I. INTRODUCTION

The fundamental equations of physics such as Maxwell equations, massless Dirac equation and massless Klein-Gordon equation are invariant with respect to conformal transformations of the metric [2]. However, the Einstein equations are not invariant with respect to these transformations. Modifications of these equations which involve the *conformal coupling* of a metric to a scalar field which leads to conformal invariance are called conformal relativity and the examples of such a theory have been studied [3–9]. Recently, in one of the versions of these theories the geometrical evolution of the universe was reinterpreted as an evolution of mass represented by a scalar field in a flat universe [10–13, 17]. The idea is quite interesting and it may help to resolve the problem of the dark energy in the universe [14–16]. Similar ideas have been developed in yet another modification of general relativity called Self Creation Cosmology [18] in which the dark energy problem together with a series of other cosmological problems including Pioneer spacecraft puzzle [19] have been studied. In yet another proposal, a conformal transformation helps to suppress the cosmological constant in the conformal frame [20].

The simplest example of a conformally invariant theory is the Brans-Dicke theory with Brans-Dicke parameter $\omega = -3/2$ [13]. Amazingly, the recent fit to supernovae data [14] shows that, despite local gravitational

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tests which give the constraint $\omega > 1000$, supernovae favour exactly the value of $\omega = -3/2$ [1]. It is also interesting to note that, apart from all the above, the Brans-Dicke theory [21] with $\omega = -3/2$ gives a border line between standard scalar field models and ghost/phantom models in the Einstein frame [22].

In this paper we discuss the exact conformal relativistic solutions, i.e., $\omega = -3/2$ Brans-Dicke solutions of isotropic Friedmann, anisotropic Kantowski-Sachs, axisymmetric Bianchi I, and Bianchi III type. We derive them from a more general context of Brans-Dicke theory taking the limit $\omega = -3/2$ and compare them with analogous solutions in low-energy superstring cosmology [23–26].

The paper is organized as follows. In Section II we discuss the status of $\omega = -3/2$ Brans-Dicke theory as conformal relativity. In Section III we present isotropic conformal cosmology solutions in the Jordan frame and then in the Einstein frame. We apply various time parametrizations in order to look for the one which is non-singular in the $\omega = -3/2$ limit. In Section IV we find anisotropic conformal cosmology solutions of Kantowski-Sachs type in the Jordan frame and present them directly in terms of the cosmic time coordinate instead of the parametric time as it was given in the previous literature. In Section V we give our conclusions.

II. CONFORMAL RELATIVITY AS $\omega = -3/2$ BRANS-DICKE THEORY

Suppose that we have two spacetime manifolds $\mathcal{M}, \tilde{\mathcal{M}}$ with metrics $g_{\mu\nu}, \tilde{g}_{\mu\nu}$ and *the same* coordinates x^μ . We say that the two manifolds are *conformal* to each other if they are related by the following *conformal transformation*

$$\tilde{g}_{\mu\nu} = \Omega^2(x)g_{\mu\nu} , \quad (\text{II.1})$$

and $\Omega(x)$ which is called a conformal factor must be a twice-differentiable function of coordinates x^μ and lie in the range $0 < \Omega < \infty$. The conformal transformations shrink or stretch the distances between the two points described by the same coordinate system x^μ on the manifolds $\mathcal{M}, \tilde{\mathcal{M}}$, respectively, but they preserve the angles between vectors (in particular, between null vectors which define the light cones) which leads to a conservation of the (global) causal structure of the manifold [33]. This means that null geodesics are left intact while the timelike geodesics are modified by conformal transformations. If we take $\Omega = \text{const.}$ we deal with the so-called *scale transformations* [34]. In fact, conformal transformations are *localized* scale transformations $\Omega = \Omega(x)$.

On the other hand, the *coordinate transformations* $x^\mu \rightarrow \tilde{x}^\mu$ only relabel the coordinates and do not change geometry and they are entirely *different* from conformal transformations [7]. This is crucial since conformal transformations lead to a *different physics* on conformally related manifolds $\mathcal{M}, \tilde{\mathcal{M}}$ [34]. Since this will usually be related to a *different coupling* of a physical field to gravity we will be talking about different *frames* in which the physics is studied (see also Ref. [35] for a slightly different view).

In this paper we discuss the following conformally invariant action [5, 8, 13]

$$\tilde{S} = \frac{1}{16\pi} \frac{1}{2} \int d^4x \sqrt{-\tilde{g}} \tilde{\Phi} \left(\frac{1}{6} \tilde{R} \tilde{\Phi} - \tilde{\square} \tilde{\Phi} \right) . \quad (\text{II.2})$$

It composes of the scalar field (the dilaton) which is conformally coupled to the metric (the graviton). Conformal invariance means that the application of the conformal transformation (II.1) with the conformal factor chosen to be

$$\tilde{\Phi} = \Omega^{-1} \Phi , \quad (\text{II.3})$$

brings (II.2) to the same form, i.e.,

$$S = \frac{1}{16\pi} \frac{1}{2} \int d^4x \sqrt{-g} \Phi \left(\frac{1}{6} R \Phi - \square \Phi \right) , \quad (\text{II.4})$$

where all the quantities have no tildes.

We do not admit any matter part into the action (or the matter energy-momentum tensor into the field equations) so that, despite we have a dilaton field, we formally deal with vacuum field equations. However, they do not look like vacuum field equations since they are obtained as a result of a non-minimal coupling of the dilaton to the graviton.

These actions (II.2) and (II.4) are usually represented in a different form by the application of the expression for a covariant d’Alembertian for a scalar field in general relativity

$$\tilde{\square} \tilde{\Phi} = \frac{1}{\sqrt{-\tilde{g}}} \tilde{\partial}_\mu \left(\sqrt{-\tilde{g}} \tilde{\partial}^\mu \tilde{\Phi} \right) , \quad (\text{II.5})$$

which after integrating out the boundary term, gives [34]

$$\tilde{S} = \frac{1}{16\pi} \frac{1}{2} \int d^4x \sqrt{-\tilde{g}} \left[\frac{1}{6} \tilde{R} \tilde{\Phi}^2 + \tilde{\partial}_\mu \tilde{\Phi} \tilde{\partial}^\mu \tilde{\Phi} \right] , \quad (\text{II.6})$$

and the second term is just a kinetic term for the scalar field (cf. [2, 33]). The equations (II.6) are of course also conformally invariant, since the application of the formulas (A.1), (A.8) and (II.3) together with the appropriate integration of the boundary term gives the same form of the equations

$$S = \frac{1}{16\pi} \frac{1}{2} \int d^4x \sqrt{-g} \left[\frac{1}{6} R \Phi^2 + \partial_\mu \Phi \partial^\mu \Phi \right] . \quad (\text{II.7})$$

In fact, due to the type of non-minimal coupling of gravity to a scalar field $\tilde{\Phi}$ or Φ in (II.6) and (II.7) and the relation to Brans-Dicke theory we say that these equations are presented in the *Jordan frame* [34, 36].

It is worth noticing that adding the self-interacting scalar field potential

$$\tilde{U}(\tilde{\Phi}) = \frac{\tilde{\lambda}}{4} \tilde{\Phi}^4 , \quad (\text{II.8})$$

with the coupling constant $\tilde{\lambda}$ is conformally-invariant (but only in $D = 4$ spacetime dimensions [13]). In order to see this, we start with the action with self-interaction potential which under conformal transformation changes as

$$\begin{aligned} S &= \frac{1}{16\pi} \frac{1}{2} \int d^4x \sqrt{-\tilde{g}} \left[\frac{1}{6} \tilde{R} \tilde{\Phi}^2 + \tilde{\partial}_\mu \tilde{\Phi} \tilde{\partial}^\mu \tilde{\Phi} + \frac{\tilde{\lambda}}{4} \tilde{\Phi}^4 \right] \\ &= \frac{1}{16\pi} \frac{1}{2} \int d^4x \sqrt{-g} \left[\frac{1}{6} R \Phi^2 + \partial_\mu \Phi \partial^\mu \Phi + \frac{\tilde{\lambda}}{4} \Phi^4 \right] , \end{aligned} \quad (\text{II.9})$$

so that the new self-interaction potential reads as

$$U(\Phi) = \frac{\tilde{\lambda}}{4} \Phi^4 . \quad (\text{II.10})$$

This fact was used in Ref. [20] where in one of the frames the cosmological constant related to Anti-deSitter solution was suppressed due to the quantum arguments in the flat Minkowski second frame.

The conformally invariant actions (II.2) and (II.4) are the basis to derive the equations of motion via the variational principle. The equations of motion for scalar fields $\tilde{\Phi}$ and Φ are conformally invariant

$$\left(\tilde{\square} - \frac{1}{6} \tilde{R} \right) \tilde{\Phi} = \Omega^{-3} \left(\square - \frac{1}{6} R \right) \Phi = 0 , \quad (\text{II.11})$$

and they have the structure of the Klein-Gordon equation with the mass term replaced by the curvature term [5]. In fact, this leads to a violation of the strong equivalence principle which may either be constrained by observations or admitted in the very early universe. The conformally invariant Einstein equations are obtained from variation of \tilde{S} with respect to the metric $\tilde{g}_{\mu\nu}$ and read as

$$\left(\tilde{R}_{\mu\nu} - \frac{1}{2} \tilde{g}_{\mu\nu} \tilde{R} \right) \frac{1}{6} \tilde{\Phi}^2 + \frac{1}{6} \left[4 \tilde{\Phi}_{;\mu} \tilde{\Phi}_{;\nu} - \tilde{g}_{\mu\nu} \tilde{\Phi}_{;\alpha} \tilde{\Phi}^{;\alpha} \right] + \frac{1}{3} \left[\tilde{g}_{\mu\nu} \tilde{\Phi} \tilde{\square} \tilde{\Phi} - \tilde{\Phi} \tilde{\Phi}_{;\mu\nu} \right] = 0 . \quad (\text{II.12})$$

Applying (A.6), (A.8), (II.3) and (A.12) into (II.12) gives the same *conformally invariant* form of the field equations as

$$\left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) \frac{1}{6} \Phi^2 + \frac{1}{6} \left[4 \Phi_{;\mu} \Phi_{;\nu} - g_{\mu\nu} \Phi_{;\alpha} \Phi^{;\alpha} \right] + \frac{1}{3} \left[g_{\mu\nu} \Phi \square \Phi - \Phi \Phi_{;\mu\nu} \right] = 0 . \quad (\text{II.13})$$

These are exactly the same field equations as in the Hoyle-Narlikar theory [7] (see also Canuto-Hsieh theory [4]). Note that the scalar field equations of motion (II.11) can be obtained by an appropriate contraction of the equations (II.12) and (II.13), so that they are not independent and do not supply any additional information [13].

We can easily relate conformal relativity to Brans-Dicke theory using conformally invariant actions (II.2) and (II.4) in the Jordan frame by defining new scalar fields $\phi, \tilde{\phi}$ as

$$\frac{1}{12}\Phi^2 = e^{-\phi}, \quad \frac{1}{12}\tilde{\Phi}^2 = e^{-\tilde{\phi}}, \quad e^{-\tilde{\phi}/2} = \Omega^{-1}e^{-\phi/2}, \quad (\text{II.14})$$

which gives these conformally invariant actions in the form

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} e^{-\phi} \left[R + \frac{3}{2} \partial_\mu \phi \partial^\mu \phi \right], \quad (\text{II.15})$$

$$\tilde{S} = \frac{1}{16\pi} \int d^4x \sqrt{-\tilde{g}} e^{-\tilde{\phi}} \left[\tilde{R} + \frac{3}{2} \tilde{\partial}_\mu \tilde{\phi} \tilde{\partial}^\mu \tilde{\phi} \right]. \quad (\text{II.16})$$

These actions, however, are special cases of the Brans-Dicke action written down in terms of the scalar field ϕ , i.e.,

$$\Phi_{BD} = e^{-\phi} \quad (\text{II.17})$$

where Φ_{BD} is the Brans-Dicke field [13, 21]. This Brans-Dicke action which is *not* conformally invariant reads as

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} e^{-\phi} [R - \omega \partial_\mu \phi \partial^\mu \phi] \quad (\text{II.18})$$

which, in view of the equations (II.15) and (II.18), shows the equivalence of the vacuum conformal relativity with Brans-Dicke theory provided that the Brans-Dicke parameter

$$\omega = -\frac{3}{2}. \quad (\text{II.19})$$

On the other hand, if one takes

$$\omega = -1 \quad (\text{II.20})$$

in (II.18), then one obtains the low-energy-effective superstring action (which is also *not* conformally invariant) for only graviton and dilaton in the spectrum [25, 26]

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} e^{-\phi} [R + \partial_\mu \phi \partial^\mu \phi]. \quad (\text{II.21})$$

In fact, the action (II.18) represents Brans-Dicke theory in a special frame which is known as *string frame* or Jordan frame. It is because in superstring theory the coupling constant g_s is related to the vacuum expectation value of the dilaton by [26]

$$g_s \propto e^{\phi/2}. \quad (\text{II.22})$$

The field equations which are obtained by the variation of (II.18) with respect to the dilaton ϕ and the graviton $g_{\mu\nu}$, respectively, are [37]

$$R + \omega \partial_\mu \phi \partial^\mu \phi - 2\omega \square \phi = 0, \quad (\text{II.23})$$

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = (\omega + 1) \partial_\mu \phi \partial_\nu \phi - \left(\frac{\omega}{2} + 1 \right) g_{\mu\nu} \partial_\rho \phi \partial^\rho \phi + g_{\mu\nu} \square \phi - \phi_{;\mu\nu}. \quad (\text{II.24})$$

It is interesting to note that the Ricci tensor which can be calculated from (II.23)-(II.24) as

$$R_{\mu\nu} = -\phi_{;\mu\nu} + (\omega + 1) (\partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \partial_\rho \phi \partial^\rho \phi + g_{\mu\nu} \square \phi), \quad (\text{II.25})$$

and for low-energy-effective superstring theory $\omega = -1$, the whole lot of its terms vanish. However, this is not the case in conformal relativity $\omega = -(3/2)$, for which this expression is not so simple.

Further on, we will look for the graviton-dilaton solutions of the most general Brans-Dicke action (II.18) whose field equations are given by [24, 25, 37]

$$R + \omega \partial_\mu \phi \partial^\mu \phi - 2\omega \square \phi = 0, \quad (\text{II.26})$$

$$R_{\mu\nu} + \phi_{;\mu\nu} - (\omega + 1) (\partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \partial_\rho \phi \partial^\rho \phi + g_{\mu\nu} \square \phi) = 0, \quad (\text{II.27})$$

taking the conformal relativity limit $\omega = -3/2$ in the end. The Kantowski-Sachs type of solutions of (II.26) and (II.27) for the common sector of superstring theories (including the axion field) were given in Ref. [24].

III. CONFORMAL FRIEDMANN COSMOLOGY

We discuss Friedmann cosmology in the two conformally related frames as given in (II.1), i.e.,

$$d\tilde{s}^2 = -d\tilde{t}^2 + \tilde{a}^2 \left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \right) \quad (\text{III.1})$$

$$ds^2 = -dt^2 + a^2 \left(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 \right), \quad (\text{III.2})$$

and $k = 0, \pm 1$ is the spatial curvature index. From (II.1), (III.1) and (III.2) one can easily see that the time coordinates and scale factors are related by [24, 37, 39]

$$d\tilde{t} = \Omega dt, \quad (\text{III.3})$$

$$\tilde{a} = \Omega a, \quad (\text{III.4})$$

where for the full conformal invariance one has to apply the definition of conformal factor (II.3). Note that there is a sign choice freedom in the equations (III.3)-(III.4) as a consequence of the conformal equivalence of the two metrics (III.1) and (III.2).

In the string frame we use the Friedmann metric (III.2) which imposed into the equations (II.26)-(II.27) for an arbitrary value of the parameter ω gives the following set of equations

$$\dot{\phi} - 3\frac{\dot{a}}{a} = \frac{\ddot{\phi}}{\phi}, \quad (\text{III.5})$$

$$-3\frac{\dot{a}^2 + k}{a^2} = -\left(\frac{\omega}{2} + 1\right)\dot{\phi}^2 + \ddot{\phi}, \quad (\text{III.6})$$

$$-2\frac{\ddot{a}}{a} - \frac{\dot{a}^2 + k}{a^2} = \frac{\omega}{2}\dot{\phi}^2 + \frac{\dot{a}}{a}\dot{\phi}. \quad (\text{III.7})$$

These equations (III.5)-(III.7) for the flat $k = 0$ Friedmann metric give the following solutions

$$a(t) = |t|^{\frac{3(\omega+1) \pm \sqrt{3(2\omega+3)}}{3(3\omega+4)}}, \quad (\text{III.8})$$

$$\phi(t) = \frac{-1 \pm \sqrt{3(2\omega+3)}}{3\omega+4} \ln |t|, \quad (\text{III.9})$$

where following pre-big-bang/ekpyrotic scenario [41–45] the solutions for negative times are also admitted. From (III.10)-(III.11) one can first find the pre-big-bang solutions for $\omega = -1$ [26] which are very well-known and read

$$a(t) = |t|^{\pm \frac{1}{\sqrt{3}}}, \quad (\text{III.10})$$

$$\phi(t) = (\pm\sqrt{3} - 1) \ln |t|. \quad (\text{III.11})$$

However, the conformal relativity solutions for $\omega = -\frac{3}{2}$ are (see also [31])

$$a(t) = |t|, \quad (\text{III.12})$$

$$\phi(t) = 2 \ln |t|, \quad (\text{III.13})$$

and show that they do not allow for two branches ‘+’ and ‘−’ (see Figs. 1 and 2) and so they do not allow the scale factor duality [25]

$$a(t) \rightarrow \frac{1}{a(-t)}, \quad \phi \rightarrow \phi - 6 \ln a, \quad (\text{III.14})$$

which is a cosmological consequence of string duality symmetries [23]. However, unlike pre-big-bang solutions (III.10)-(III.11) which must be regularized at Big-Bang singularity because both the curvature and the string coupling (II.22) diverge there, the solutions (III.12)-(III.13) do not lead to strong coupling singularity in the sense of string theory, since the string coupling constant

$$g_s = e^{\frac{\phi}{2}} = |t|, \quad (\text{III.15})$$

is regular for $t = 0$. This has an interesting analogy with the ekpyrotic/cyclic universe scenario where, in fact, the transition through Big-Bang singularity takes place in the weak coupling regime [45].

Now let us discuss the isotropic Friedmann $k = \pm 1$ solutions of the system (III.5)-(III.7). After introducing a new time parameter

$$\zeta = \int \frac{dt}{a(t)}, \quad (\text{III.16})$$

the solutions for $k = +1$ are [27]

$$a(\zeta) = (\sin \zeta)^{\frac{1-\sigma}{2}} (\cos \zeta)^{\frac{1+\sigma}{2}}, \quad (\text{III.17})$$

$$\phi(\zeta) = -\sigma \ln(\tan \zeta), \quad (\text{III.18})$$

while the solutions for $k = -1$ are

$$a(\zeta) = (\sinh \zeta)^{\frac{1-\sigma}{2}} (\cosh \zeta)^{\frac{1+\sigma}{2}}, \quad (\text{III.19})$$

$$\phi(\zeta) = -\sigma \ln(\tanh \zeta), \quad (\text{III.20})$$

where

$$\sigma = \pm \frac{3}{3 + 2\omega}. \quad (\text{III.21})$$

From the definition of the parameter σ in (III.21) one can see that in the conformal relativistic limit $\omega = -3/2$ this parameter diverges, i.e., $\sigma \rightarrow \pm\infty$ and consequently the solutions (III.17) and (III.19) are inappropriate. Apparently, the status of the $\omega = -3/2$ case has not been fully cleared out so far. In particular, this case was never expressed in terms of cosmic time instead of parametric (generalized conformal) time, although it was presumably solved in earlier references [28–30].

In this context we will then discuss the conformal relativistic solutions of the system (III.5)-(III.7) for $\omega = -3/2$ (see also [31])

$$\dot{\phi} - 3\frac{\dot{a}}{a} = \frac{\ddot{\phi}}{\phi}, \quad (\text{III.22})$$

$$-3\frac{\dot{a}^2 + k}{a^2} = -\frac{1}{4}\dot{\phi}^2 + \ddot{\phi}, \quad (\text{III.23})$$

$$-2\frac{\ddot{a}}{a} - \frac{\dot{a}^2 + k}{a^2} = -\frac{3}{4}\dot{\phi}^2 + \frac{\dot{a}}{a}\dot{\phi}. \quad (\text{III.24})$$

Using a new time coordinate [24]

$$dt = a^3 e^{-\phi} d\tau, \quad (\text{III.25})$$

the equation (III.22) reads as

$$\phi_{,\tau\tau} = 0, \quad (\text{III.26})$$

where $(\dots)_{,\tau}$ describes a derivative with respect to τ . The equations (III.23) and (III.24) now are

$$-3\frac{a_{,\tau}^2}{a^2} - 3ka^4 e^{-2\phi} = \frac{3}{4}\phi_{,\tau}^2 - 3\frac{a_{,\tau}}{a}\phi_{,\tau} + \phi_{,\tau\tau}, \quad (\text{III.27})$$

$$-2\frac{a_{,\tau\tau}}{a} + 5\frac{a_{,\tau}^2}{a^2} - 3\frac{a_{,\tau}}{a}\phi_{,\tau} - ka^4 e^{-2\phi} = -\frac{3}{4}\phi_{,\tau}^2. \quad (\text{III.28})$$

The sum of (III.27) and (III.28) gives

$$(\ln a)_{,\tau\tau} + 2ka^4 e^{-2\phi} = 0. \quad (\text{III.29})$$

or

$$(\ln M)_{,\tau\tau} + 8kM = 0, \quad (\text{III.30})$$

where

$$M(\tau) = a^4 e^{-2\phi} . \quad (\text{III.31})$$

The solution of the equation (III.30) reads as

$$\frac{1}{\sqrt{M(\tau)}} = \cosh \beta \tau + \sqrt{1 - \frac{4k}{\beta^2}} \sinh \beta \tau , \quad (\text{III.32})$$

where $\beta = \text{const.}$ and $\beta^2/4 > k$ which suggests that the only possibility in order not to restrict the values of β is to admit $k = -1$. Note that (III.32) can also be expressed as

$$\frac{1}{\sqrt{M(\tau)}} = \cosh \beta \tau + \frac{\sinh \beta \tau}{|\beta| \tau} \tau \sqrt{\beta^2 - 4k} . \quad (\text{III.33})$$

From (III.26) we have that

$$\phi(\tau) = \alpha \tau + \gamma , \quad (\text{III.34})$$

and without a loss of generality taking $\gamma = 0$ we have

$$a(\tau) = \frac{e^{\frac{\alpha}{2}\tau}}{\left[\cosh \beta \tau + \sqrt{1 - \frac{4k}{\beta^2}} \sinh \beta \tau \right]^{\frac{1}{2}}} . \quad (\text{III.35})$$

In order to deparametrize the solution (III.35) one should use (III.25), i.e.,

$$t(\tau) = \int M^{3/4} e^{\phi/2} d\tau = \int \frac{e^{\frac{\alpha}{2}\tau}}{\left(\cosh \beta \tau + \sqrt{1 - \frac{4k}{\beta^2}} \sinh \beta \tau \right)^{\frac{3}{2}}} d\tau . \quad (\text{III.36})$$

Notice that by using the definition (III.31) the equations (III.27) and (III.28) read as

$$\left(\frac{M_{,\tau}}{M} \right)^2 + 16kM = 0 , \quad (\text{III.37})$$

$$\frac{11}{16} \left(\frac{M_{,\tau}}{M} \right)^2 - \frac{1}{2} \frac{M_{,\tau\tau}}{M} - kM = 0 , \quad (\text{III.38})$$

It seems that the parametrization (III.25) is also a bit awkward. In fact, by putting (III.32) into the constraints (III.37) and (III.38) from both of them one gets a very restrictive condition on the solution (III.32) such as

$$\beta^2 \left[\cosh \beta \tau + \sqrt{1 - \frac{4k}{\beta^2}} \sinh \beta \tau \right]^2 = 0 . \quad (\text{III.39})$$

Without the requirement of restricting the values of the cosmic time this condition necessarily requires that the constant

$$\beta = 0 . \quad (\text{III.40})$$

This, on the other hand, in the limit $\beta \rightarrow 0$ gives from (III.32) that

$$\frac{1}{\sqrt{M(\tau)}} = 1 + 2\sqrt{-k}\tau , \quad (\text{III.41})$$

(so that this solution holds only for $k = -1$ models) which then from (III.31) gives

$$a(\tau) = \frac{e^{\frac{\alpha}{2}\tau}}{(1 + 2\sqrt{-k}\tau)^{1/2}} , \quad (\text{III.42})$$

$$\phi(\tau) = \alpha \tau . \quad (\text{III.43})$$

From the form of the above solutions one can immediately see that there exists only the solution for negative curvature $k = -1$ Friedmann models and that the solutions for $k = +1$ is not admissible at all within the framework of conformal cosmology. It is advisable to notice that the equations (III.37)-(III.38) are equivalent to

$$M_{,\tau\tau} + 24kM^2 = 0 , \quad (III.44)$$

$$M_{,\tau}^2 + 16kM^3 = 0 . \quad (III.45)$$

These equations suggest an appropriate change of time coordinate as

$$d\tau = \frac{d\eta}{2\sqrt{M}} , \quad (III.46)$$

which transfers them into an easy to integrate form

$$M_{,\eta\eta} + 4kM = 0 , \quad (III.47)$$

$$M_{,\eta}^2 + 4kM^2 = 0 . \quad (III.48)$$

The solution of the system (III.47)-(III.48) is very straightforward and reads as

$$M = M_0 e^{\pm 2\sqrt{-k}\eta} . \quad (III.49)$$

On the other hand, the Eq. (III.26) in terms of η -time reads as

$$\left[\ln(\sqrt{M}\phi_{,\eta}) \right]_{,\eta} = 0 , \quad (III.50)$$

which solves by

$$\phi(\eta) = \frac{c_1}{\mp 2\sqrt{-k}} e^{\mp 2\sqrt{-k}\eta} + c_2 , \quad (III.51)$$

where c_1, c_2 are constants. Using this, one has for the scale factor

$$a(\eta) = e^{\phi/2} M^{\frac{1}{4}} = \exp\left(\frac{c_1}{\mp 2\sqrt{-k}} e^{\mp 2\sqrt{-k}\eta} + c_2\right) M_0^{1/4} e^{\pm \frac{1}{2}\sqrt{-k}\eta} . \quad (III.52)$$

In order to find the status of the solution (III.42) and (III.43) in the Jordan frame, i.e., the solution of the Brans-Dicke theory with $\omega = -3/2$, we now use the definition of the conformal factor (II.3) in terms of the fields ϕ and $\tilde{\phi}$ as follows

$$\Omega = \frac{e^{-\phi/2}}{e^{-\tilde{\phi}/2}} , \quad (III.53)$$

and make an appropriate transformation into the Einstein frame in which the scalar field is minimally coupled to gravity. This can be achieved by the assumption that one of the scalar fields is constant. Let us assume that

$$\tilde{\phi} = \tilde{\phi}_0 = \text{const.} \quad (III.54)$$

is such a field, which means that the conformal transformation from the conformal (or Jordan/string) frame to the Einstein frame reads as

$$\Omega_E = e^{\tilde{\phi}_0/2} e^{-\phi/2} , \quad (III.55)$$

and all the quantities in the Einstein frame will then be labeled by tildas. The conformal transformation then applied to (III.3) and (III.4) with the help of (III.25) gives

$$d\tilde{t} = a^3 e^{-\frac{3}{2}\phi} e^{\tilde{\phi}_0/2} d\tau , \quad (III.56)$$

which, after the application of (III.42) and (III.43) reads as

$$d\tilde{t} = \frac{e^{\tilde{\phi}_0/2}}{(1 + 2\sqrt{-k}\tau)^{3/2}} d\tau . \quad (III.57)$$

Then, it produces the relation between the Einstein frame time \tilde{t} and the τ -time as

$$\tilde{t} - \tilde{t}_0 = -\frac{e^{\tilde{\phi}_0/2}}{\sqrt{-k}\sqrt{1+2\sqrt{-k}\tau}}, \quad (\text{III.58})$$

where $\tilde{t}_0 = \text{const.}$ Using (III.4) and (III.58) we have in the Einstein frame

$$\tilde{a}(\tilde{t}) = \sqrt{-k}(\tilde{t}_0 - \tilde{t}). \quad (\text{III.59})$$

Due to the sign choice freedom in (III.3)-(III.4) and taking without the loss of generality $\tilde{t}_0 = 0$ one can write down this solution as

$$\tilde{a}(\tilde{t}) = \sqrt{-k} |\tilde{t}|. \quad (\text{III.60})$$

This, on the other hand, is just the Milne model which is equivalent to Minkowski space. In order to check whether it is consistent let us just study this problem starting directly from the Brans-Dicke theory in the Einstein frame.

The Brans-Dicke action in the Einstein frame reads as (see e.g. [13])

$$\tilde{S} = \frac{1}{16\pi} \int d^4x \sqrt{-\tilde{g}} \left[\tilde{R} - \left(\omega + \frac{3}{2} \right) \tilde{\partial}_\mu \phi \tilde{\partial}^\mu \phi \right] \quad (\text{III.61})$$

which for $\omega = -3/2$ gives exactly the Einstein-Hilbert action (no matter energy momentum tensor). The resulting Einstein frame equations for (III.1) are [13]

$$\left(\omega + \frac{3}{2} \right) \left[\ddot{\phi} + 3 \frac{\dot{\tilde{a}}}{\tilde{a}} \dot{\phi} \right] = 0, \quad (\text{III.62})$$

$$3 \frac{\dot{\tilde{a}}^2 + k}{\tilde{a}^2} = \left(\omega + \frac{3}{2} \right) \dot{\phi}^2, \quad (\text{III.63})$$

$$-2 \frac{\ddot{\tilde{a}}}{\tilde{a}} - \frac{\dot{\tilde{a}}^2 + k}{\tilde{a}^2} = \left(\omega + \frac{3}{2} \right) \dot{\phi}^2, \quad (\text{III.64})$$

where the dot in these equations represents a differentiation with respect to \tilde{t} . The solutions of (III.62)-(III.64) for an arbitrary value of the parameter $\omega \neq -3/2$ and $k = 0$ read as

$$\tilde{a} = |\tilde{t}|^{\frac{1}{3}}, \quad (\text{III.65})$$

$$\phi = \phi_0 + \frac{1}{\sqrt{3(\omega + \frac{3}{2})}} \ln |\tilde{t}|. \quad (\text{III.66})$$

First notice that for $\omega = -3/2$ and $k = 0$ the unique solution gives

$$\dot{\tilde{a}} = 0, \phi - \text{arbitrary}, \quad (\text{III.67})$$

which is just a flat Minkowski universe. This claim seems to be consistent with our solution (III.12) in the Jordan frame. Finally, for the case of our interest, $k \neq 0$, we get from (III.63) that

$$\tilde{a} = \sqrt{-k} |\tilde{t}|, \phi - \text{arbitrary}, \quad (\text{III.68})$$

which is admissible only for $k = -1$ and this solution represents Milne universe [7] (in which there is no acceleration of the expansion since the deceleration parameter $q = \ddot{\tilde{a}}\tilde{a}/\dot{\tilde{a}}^2 = 0$). However, its relation to Minkowski spacetime

$$dS^2 = -dT^2 + dR^2 + R^2 d\theta^2 + R^2 \sin^2 \theta d\varphi^2 \quad (\text{III.69})$$

requires coordinate transformation

$$T = \tilde{t} \sqrt{1+r^2}, \quad R = \tilde{t} r, \quad (\text{III.70})$$

which involves the two time scales - a dynamical one \tilde{t} and an atomic one T [7, 10] which may be responsible for the cosmological redshift effect. On the other hand, the solution for $k = +1$ would be possible only if

the cosmological constant was admitted - again, cosmological redshift in this Static Einstein model would be the result of a different time scaling [7]. One should also notice that it is easy to add the cosmological constant term into the action (III.61) (which is equivalent to self-interaction potential with $\tilde{\Phi} = \text{const.}$ in (II.9)). This would allow for a non-flat Anti-deSitter (or a deSitter) solution as in Ref. [20], which would then be transformed into the Jordan frame with $\Phi \neq \text{const.}$

In conclusion, it seems that the reason for having only the flat solutions in $\omega = -3/2$ Brans-Dicke cosmology is that in the Einstein frame action (III.61) in this limit, the kinetic term of the scalar field vanishes, and the action is equivalent to a vacuum Einstein-Hilbert action, which necessarily admits only vacuum (i.e. flat) solutions.

On the other hand, the time scaling of the scale factor for the Milne model is the same as the scaling for the cosmological fluid of cosmic strings $p = -(1/3)\rho$ which have negative pressure. This fact seems to be consistent with the supernovae data which requires negative pressure for having cosmic acceleration, although it is not strong enough to be fully consistent with phantom $p < -\rho$ matter, which is favoured with the most recent data [32].

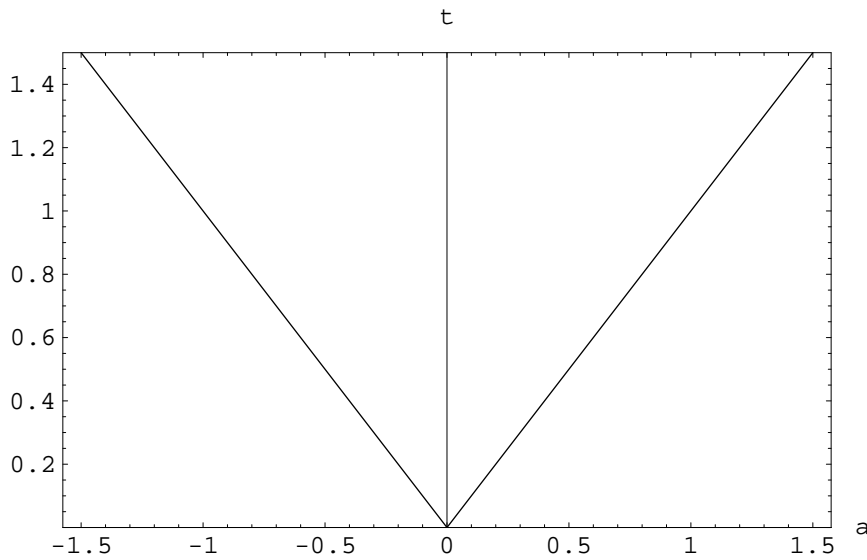


FIG. 1: The scale factor $a(t)$ (III.12) for an isotropic conformal relativity ($\omega = -3/2$) model. There is a curvature singularity (Big-Bang) at $t = 0$.

IV. CONFORMAL KANTOWSKI-SACHS COSMOLOGY

In this Section we choose an anisotropic Kantowski-Sachs form of the metric of spacetime, with [40]

$$ds^2 = -dt^2 + X^2(t)dr^2 + Y^2(t)d\Omega_k^2, \quad (\text{IV.1})$$

where the angular metric is

$$d\Omega_k^2 = d\theta^2 + S^2(\theta)d\psi^2, \quad (\text{IV.2})$$

and

$$S(\theta) = \begin{cases} \sin(\theta) & \text{for } k = +1, \\ \theta & \text{for } k = 0, \\ \sinh(\theta) & \text{for } k = -1. \end{cases} \quad (\text{IV.3})$$

The functions $X(t)$ and $Y(t)$ are the expansion scale factors. We shall consider models of all three curvatures in the same analysis, although the pure Kantowski-Sachs models are of $k = +1$ spatial curvature. For $k = 0$ we deal with axisymmetric Bianchi type I models, while for $k = -1$ we deal with Bianchi III models.

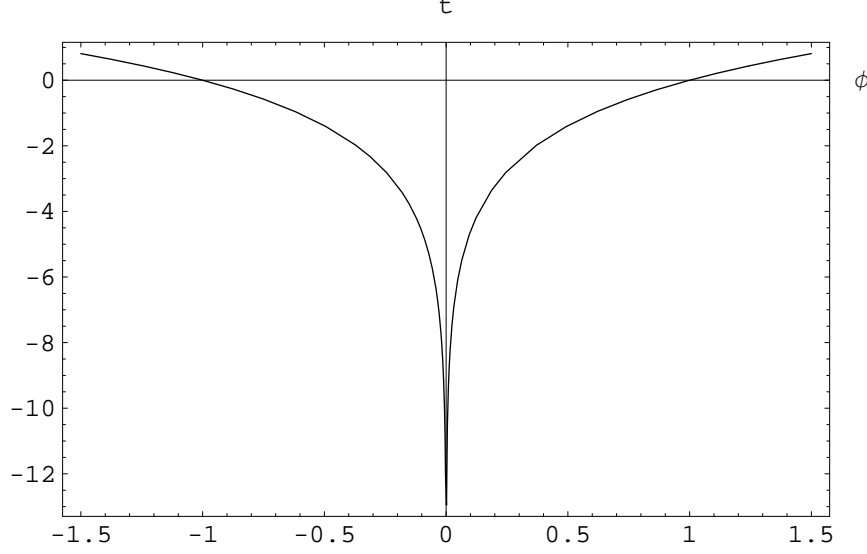


FIG. 2: The evolution of the scalar field $\phi(t)$ (III.13) in an isotropic conformal relativity ($\omega = -3/2$) model. There is no strong coupling singularity, since the string coupling constant $g_s = \exp \phi/2 \rightarrow 0$ for $t \rightarrow 0$.

After the conformal transformation (II.1) one can see that the transformed metric is

$$d\tilde{s}^2 = -d\tilde{t}^2 + \tilde{X}^2(\tilde{t})dr^2 + \tilde{Y}^2(\tilde{t})d\Omega_k^2, \quad (\text{IV.4})$$

and the time coordinates and scale factors are related via [24, 37]

$$d\tilde{t} = \Omega dt, \quad (\text{IV.5})$$

$$\tilde{X} = \Omega X, \quad (\text{IV.6})$$

$$\tilde{Y} = \Omega Y. \quad (\text{IV.7})$$

The nonzero Ricci tensor components are [24]

$$R_0^0 = \frac{\ddot{X}}{X} + 2\frac{\ddot{Y}}{Y}, \quad (\text{IV.8})$$

$$R_1^1 = \frac{\ddot{X}}{X} + 2\frac{\dot{Y}}{Y}\frac{\dot{X}}{X}, \quad (\text{IV.9})$$

$$R_2^2 = R_3^3 = \frac{k + \dot{Y}^2}{Y^2} + \frac{\ddot{Y}}{Y} + \frac{\dot{Y}}{Y}\frac{\dot{X}}{X}, \quad (\text{IV.10})$$

and the scalar curvature is

$$R = 2\frac{\ddot{X}}{X} + 4\frac{\ddot{Y}}{Y} + 2\frac{k + \dot{Y}^2}{Y^2} + 4\frac{\dot{Y}}{Y}\frac{\dot{X}}{X}. \quad (\text{IV.11})$$

The field equations (II.27) become

$$\frac{\ddot{X}}{X} + 2\frac{\ddot{Y}}{Y} - \ddot{\phi} - (\omega + 1) \left(-\frac{\dot{X}}{X}\dot{\phi} - 2\frac{\dot{Y}}{Y}\dot{\phi} - \ddot{\phi} \right) = 0, \quad (\text{IV.12})$$

$$\frac{\ddot{X}}{X} + 2\frac{\dot{Y}}{Y}\frac{\dot{X}}{X} - \frac{\dot{X}}{X}\dot{\phi} - (\omega + 1) \left(\dot{\phi}^2 - \frac{\dot{X}}{X}\dot{\phi} - 2\frac{\dot{Y}}{Y}\dot{\phi} - \ddot{\phi} \right) = 0, \quad (\text{IV.13})$$

$$\frac{k + \dot{Y}^2}{Y^2} + \frac{\ddot{Y}}{Y} + \frac{\dot{Y}}{Y} \frac{\dot{X}}{X} - \frac{\dot{Y}}{Y} \dot{\phi} - (\omega + 1) \left(\dot{\phi}^2 - \frac{\dot{X}}{X} \dot{\phi} - 2 \frac{\dot{Y}}{Y} \dot{\phi} - \ddot{\phi} \right) = 0 \quad (\text{IV.14})$$

The field equation (II.26) reads

$$2 \frac{\ddot{X}}{X} + 4 \frac{\ddot{Y}}{Y} + 2 \frac{k + \dot{Y}^2}{Y^2} + 4 \frac{\dot{Y}}{Y} \frac{\dot{X}}{X} - \omega \dot{\phi}^2 + 2\omega \ddot{\phi} + 2\omega \left(\frac{\dot{X}}{X} + 2 \frac{\dot{Y}}{Y} \right) \dot{\phi} = 0. \quad (\text{IV.15})$$

Adding the Eqs. (IV.12), (IV.13) with doubled (IV.14) and subtracting from this sum Eq. (IV.15) we get

$$\ddot{\phi} - \dot{\phi}^2 + \left(\frac{\dot{X}}{X} + 2 \frac{\dot{Y}}{Y} \right) \dot{\phi} = 0. \quad (\text{IV.16})$$

At this stage we introduce a new time coordinate τ via relation

$$dt = XY^2 e^{-\phi} d\tau. \quad (\text{IV.17})$$

Then Eq. (IV.16) becomes

$$\phi_{,\tau\tau} = 0, \quad (\text{IV.18})$$

which solves as

$$\phi(\tau) = a\tau + \gamma. \quad (\text{IV.19})$$

Using the time coordinate (IV.17) equations (IV.12)–(IV.14) become

$$\left(\frac{X_{,\tau}}{X} \right)_{,\tau} + 2 \left(\frac{Y_{,\tau}}{Y} \right)_{,\tau} - 2 \frac{Y_{,\tau}}{Y} \left(\frac{Y_{,\tau}}{Y} + 2 \frac{X_{,\tau}}{X} \right) + 2 \phi_{,\tau} \left(\frac{X_{,\tau}}{X} + 2 \frac{Y_{,\tau}}{Y} \right) + \omega \phi_{,\tau}^2 = 0, \quad (\text{IV.20})$$

$$\left(\frac{X_{,\tau}}{X} \right)_{,\tau} = 0, \quad (\text{IV.21})$$

$$\left(\frac{Y_{,\tau}}{Y} \right)_{,\tau} + k X^2 Y^2 e^{-2\phi} = 0. \quad (\text{IV.22})$$

The solution of equation (IV.21) is simply

$$X = \frac{1}{A_0} e^{c\tau}, \quad (\text{IV.23})$$

where c and A_0 are constants. When we put in the Eq. (IV.23) $A_0 = 1$ and $c = \frac{1}{2}(a + p)$, where a is taken from (IV.19) and p is a constant, we obtain (IV.23) in the following form

$$X(\tau) = e^{\frac{1}{2}(a+p)\tau} \quad (\text{IV.24})$$

In Eq. (IV.19) we set $\gamma = 0$ without loss of generality. Then from Eqs. (IV.20) and (IV.22) we obtain

$$Y(\tau) = \begin{cases} e^{\frac{1}{2}(a-p)\tau} \sqrt{\frac{1}{4}[(2\omega + 3)a^2 + p^2]} \{\cosh[\frac{1}{4}(a^2\omega + 3a^2 + p^2)]\tau\}^{-1/2} & \text{for } k = +1, \\ e^{\frac{1}{2}(a-p)\tau} e^{-\sqrt{\frac{1}{4}[(2\omega + 3)a^2 + p^2]}\tau} & \text{for } k = 0, \\ e^{\frac{1}{2}(a-p)\tau} \sqrt{\frac{1}{4}[(2\omega + 3)a^2 + p^2]} \{\sinh[\frac{1}{4}(a^2\omega + 3a^2 + p^2)]\tau\}^{-1/2} & \text{for } k = -1, \end{cases} \quad (\text{IV.25})$$

For $k = 0$ after integration of Eq. (IV.17) we get

$$t(\tau) = \frac{-2e^{-\frac{1}{2}\tau(p-a+2\sqrt{p^2+a^2(3+2\omega)})}}{(-a+p+2\sqrt{p^2+a^2(2+3\omega)})}. \quad (\text{IV.26})$$

And for $k \neq 0$ Eq. (IV.17) is integrable for $a=p$. We get

$$t(\tau) = \begin{cases} \pm \frac{a}{\sqrt{2}} \sqrt{2+\omega} \coth \left[\pm \frac{a}{\sqrt{2}} \sqrt{2+\omega} \tau \right] & \text{for } k = -1, \\ \mp \frac{a}{\sqrt{2}} \sqrt{2+\omega} \coth \left[\mp \frac{a}{\sqrt{2}} \sqrt{2+\omega} \tau \right] & \text{for } k = +1. \end{cases} \quad (\text{IV.27})$$

Then for $k = 0$ (axisymmetric Bianchi I type) we obtain solution in the form

$$X(t) = \left\{ -\frac{1}{2}t \left(-a + p + 2\sqrt{p^2 + a^2(3+2\omega)} \right) \right\}^{-\frac{a+p}{p-a+2\sqrt{p^2+a^2(3+2\omega)}}}, \quad (\text{IV.28})$$

$$Y(t) = \left\{ -\frac{1}{2}t \left(-a + p + 2\sqrt{p^2 + a^2(3+2\omega)} \right) \right\}^{\frac{p-a+\sqrt{p^2+a^2(3+2\omega)}}{p-a+2\sqrt{p^2+a^2(3+2\omega)}}}, \quad (\text{IV.29})$$

$$\phi(t) = \frac{2a}{a-p-2\sqrt{p^2+a^2(3+2\omega)}} \ln \left\{ -\frac{1}{2}t \left(p-a+2\sqrt{p^2+a^2(3+2\omega)} \right) \right\}. \quad (\text{IV.30})$$

These solutions generalize the isotropic solution given by (III.12) and (III.13). Taking $\omega = -3/2$ we get

$$X(t) = \left\{ \frac{1}{2}t(a-3p) \right\}^{\frac{a+p}{a-3p}}, \quad (\text{IV.31})$$

$$Y(t) = \left\{ \frac{1}{2}t(a-3p) \right\}^{\frac{a-2p}{a-3p}}, \quad (\text{IV.32})$$

$$\phi(t) = \frac{2a}{a-3p} \ln \left\{ \frac{1}{2}t(a-3p) \right\}. \quad (\text{IV.33})$$

The plots of these solutions for conformal relativity ($\omega = -3/2$) and for the different values of the parameters a and p are given in Figs. 3, 4, and 5.

For non-zero $k = \pm 1$ curvature we have

$$X(t) = \left\{ k \frac{\frac{a\sqrt{2+\omega}}{\sqrt{2}} + t}{\frac{a\sqrt{2+\omega}}{\sqrt{2}} - t} \right\}^{\frac{1}{\sqrt{2(2+\omega)}}} \quad (\text{IV.34})$$

$$Y(t) = \sqrt{k \left[\frac{a^2(2+\omega)}{2} - t^2 \right]} \quad (\text{IV.35})$$

$$\phi(t) = \ln \left\{ k \frac{t + \frac{a\sqrt{2+\omega}}{\sqrt{2}}}{t - \frac{a\sqrt{2+\omega}}{\sqrt{2}}} \right\}^{\frac{1}{\sqrt{2(2+\omega)}}} \quad (\text{IV.36})$$

In order to understand the nature of both initial and final singularities it is important to study the evolution of the volume

$$V(t) = X(t)Y^2(t) = \left[k \left(\frac{a\sqrt{2+\omega}}{\sqrt{2}} + t \right) \right]^{1+\frac{1}{\sqrt{2(2+\omega)}}} \left[k \left(\frac{a\sqrt{2+\omega}}{\sqrt{2}} - t \right) \right]^{1-\frac{1}{\sqrt{2(2+\omega)}}}. \quad (\text{IV.37})$$

In the case of conformal relativity $\omega = -3/2$ we have

$$X = k \frac{\frac{a}{2} + t}{\frac{a}{2} - t}, \quad (\text{IV.38})$$

$$Y^2 = k \left(\frac{a}{2} + t \right) \left(\frac{a}{2} - t \right), \quad (\text{IV.39})$$

$$\phi = \ln \left[k \frac{t + \frac{a}{2}}{t - \frac{a}{2}} \right], \quad (\text{IV.40})$$

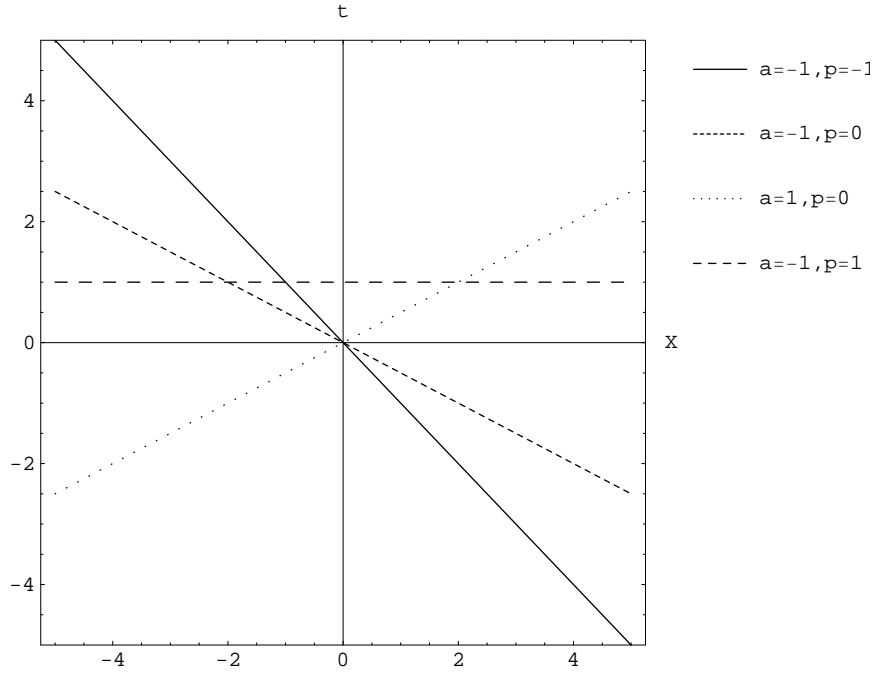


FIG. 3: The plots of the scale factor X (Eq. (IV.28)) in conformal relativity ($\omega = -3/2$) for the axisymmetric Bianchi I ($k = 0$) cosmological models. Different shapes of the plots depend on the values of the constants a and p .

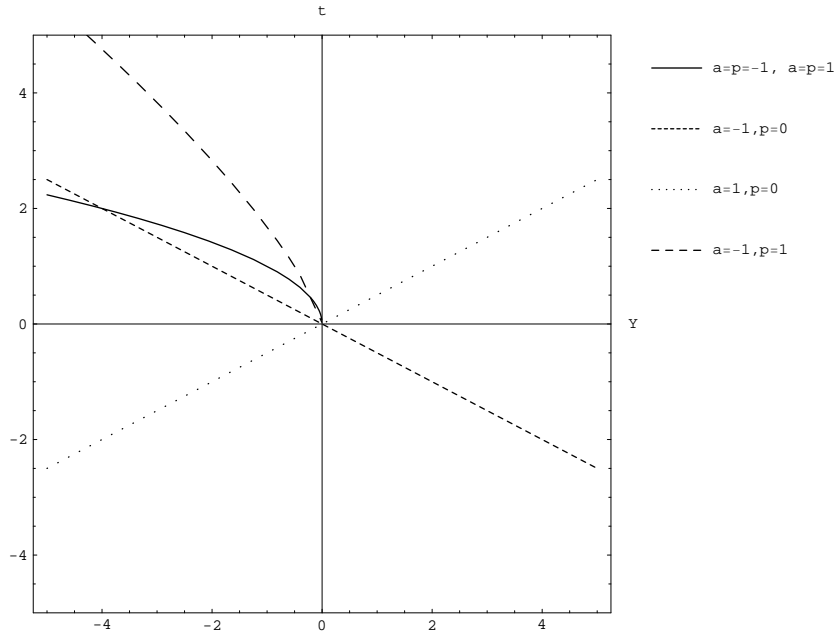


FIG. 4: The plots of the scale factor Y (Eq. (IV.29)) in conformal relativity ($\omega = -3/2$) for the axisymmetric Bianchi I ($k = 0$) cosmological models. Different shapes of the plots depend on the values of the constants a and p .

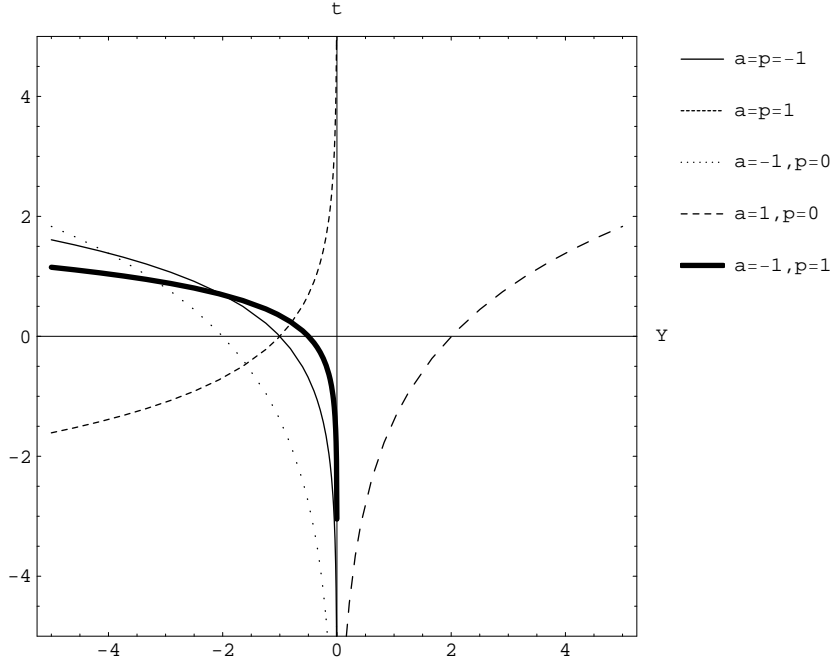


FIG. 5: The plots of the dilaton field ϕ (Eq. (IV.30)) in conformal relativity ($\omega = -3/2$) for the axisymmetric Bianchi I ($k = 0$) cosmological models. Different shapes of the plots depend on the values of the constants a and p .

where the time coordinate has the ranges

$$0 \leq t^2 \leq \frac{a^2}{4} \text{ for } k = +1, \quad (\text{IV.41})$$

$$t^2 \geq \frac{a^2}{4} \text{ for } k = -1. \quad (\text{IV.42})$$

The volume (IV.37) scales as

$$V = \left(\frac{a}{2} + t \right)^2, \quad (\text{IV.43})$$

which shows that the divergent term for $t = a/2$ was cancelled. This means we deal with initial Big-Bang type of singularity at $t = -a/2$ where $X = Y = 0$ (though it is weak coupling since $e^\phi \rightarrow 0$) while at $t = a/2$ the volume is finite despite the fact that $X \rightarrow \infty$ and $Y = 0$ there and suggests the appearance of the barrel singularity (though it is strong coupling since $e^\phi \rightarrow \infty$). The plots of the solutions (IV.34) and (IV.35) for $\omega = -3/2$ are given in Figs. 6, 7, and 8. The string cosmology case $\omega = -1$ was given in Ref. [24].

V. CONCLUSIONS

We have studied isotropic Friedmann, anisotropic Kantowski-Sachs, axisymmetric Bianchi I, and Bianchi III cosmological models within the framework of $\omega = -3/2$ Brans-Dicke cosmology which is equivalent to the so-called conformal relativity. In fact, we have started from a general class of solutions which are the vacuum Brans-Dicke theory solutions in the Jordan frame. These solutions are parametrized by the Brans-Dicke parameter ω . The conformal relativity solutions are given for $\omega = -3/2$, while the low-energy-superstring (pre-big-bang) type of solutions are given for $\omega = -1$. It emerged that the conformal relativity limit $\omega = -3/2$ is nontrivial and cannot be obtained automatically from the Brans-Dicke solutions by most of the routine Brans-Dicke time parametrizations. Despite that, we were able to find an appropriate time parametrization to show that there exist only the $k = 0$ and $k = -1$ isotropic solutions in the Jordan frame. We have also shown that the $k = -1$ solution in the Jordan frame represents the Milne universe in

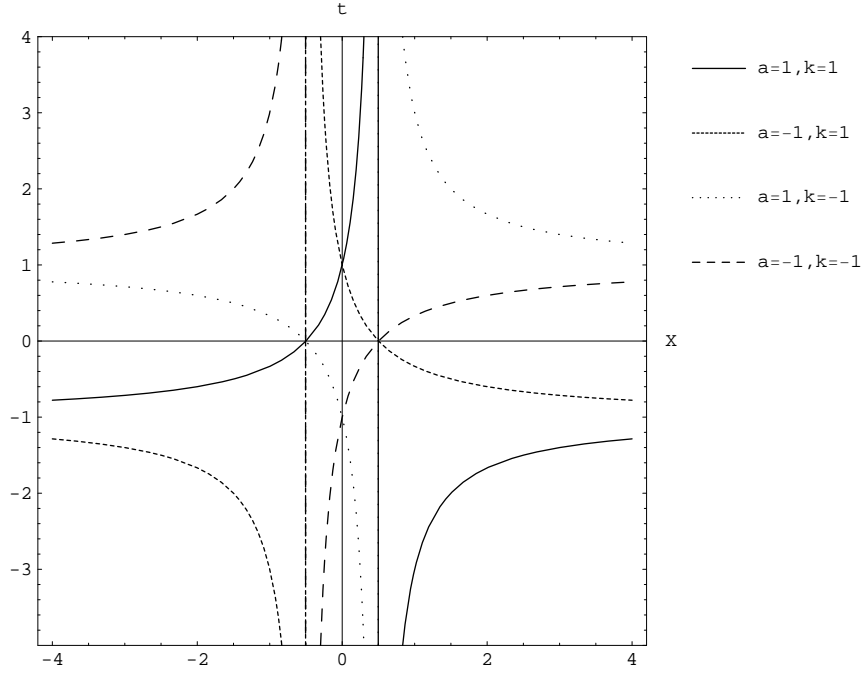


FIG. 6: The plots of the scale factor X (Eq. (IV.34)) in conformal relativity ($\omega = -3/2$) for Kantowski-Sachs ($k = +1$) and Bianchi III ($k = -1$) cosmological models. Different shapes of the plots depend on the values of the constant a .

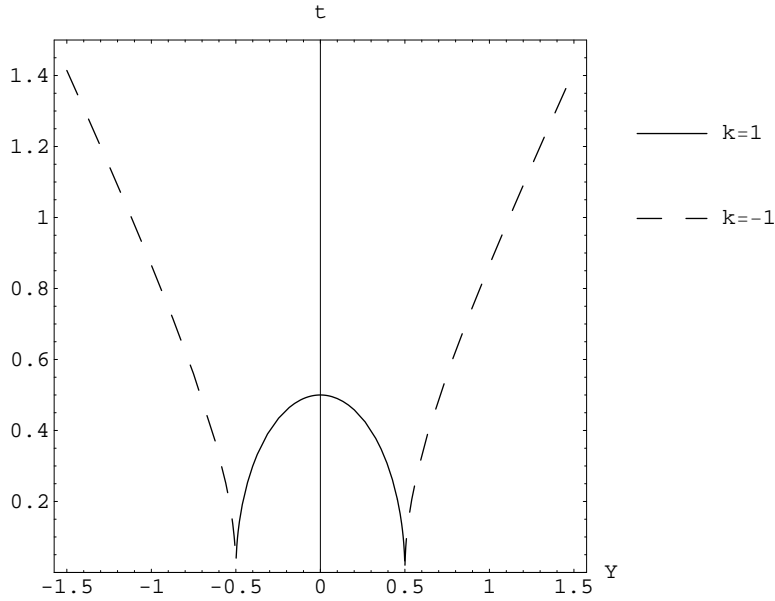


FIG. 7: The plots of the scale factor Y (Eq. (IV.35)) in conformal relativity ($\omega = -3/2$) for Kantowski-Sachs ($k = +1$) and Bianchi III ($k = -1$) cosmological models.

the conformally related (with scalar field minimally coupled to gravity) Einstein frame which, in turn, is equivalent to a flat Minkowski spacetime. Because of that, we claim that only the flat models are consistent with $\omega = -3/2$ Brans-Dicke cosmology. In a way this is not a surprise since in the Einstein frame the kinetic term of the scalar field vanishes for $\omega = -3/2$.

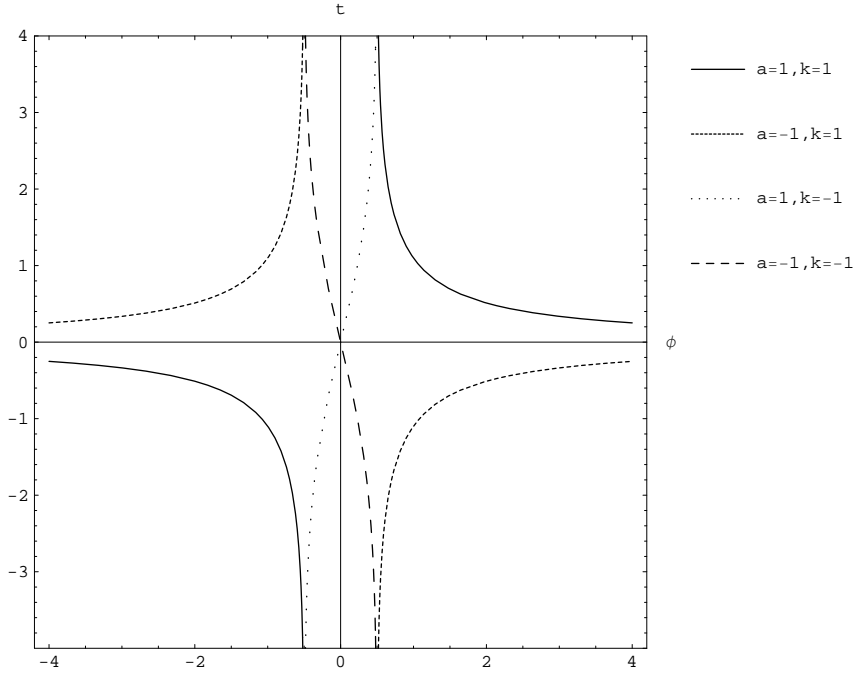


FIG. 8: The plots of the dilaton field ϕ (Eq. (IV.36)) in conformal relativity ($\omega = -3/2$) for Kantowski-Sachs ($k = +1$) and Bianchi III ($k = -1$) cosmological models. Different shapes of the plots depend on the values of the constant a .

An additional point of interest in the $\omega = -3/2$ solutions is the fact that the recent fit to supernovae data [14] shows, that despite local gravitational tests which give the constraint $\omega > 1000$, supernovae favour exactly the value of $\omega = -3/2$ [1]. Besides, $\omega = -3/2$ gives a border line between a standard scalar field model and a ghost/phantom model in the Einstein frame [21, 22]. This may be one of the crucial points for the success of the fit in the Jordan frame although the time scaling of the scale factor for the Milne universe is the same as the scaling for the cosmological fluid of cosmic strings $p = -(1/3)\rho$ in the Einstein frame which is not strong enough to be fully consistent with phantom $p < -\rho$ matter favoured by the most recent supernovae data [32].

Apart from isotropic solutions we have also studied anisotropic Kantowski-Sachs, axisymmetric Bianchi I, and Bianchi III type solutions. In particular, anisotropic Kantowski-Sachs models of non-zero spatial curvature are admissible in $\omega = -3/2$ Brans-Dicke theory, i.e., in conformal relativity. This means that an additional scale factor which appears in Kantowski-Sachs models gives an extra degree of freedom to the theory and makes it less restrictive than in an isotropic Friedmann case although these solutions should be conformally equivalent to vacuum solutions in the Einstein frame. In our paper these anisotropic solutions were fully deparametrized in terms of the cosmic time t and not given in terms of the parametric time only as in the previous literature.

Besides, in the isotropic Friedmann case, the advantage of conformal relativity solutions to pre-big-bang solutions is that there is no strong coupling singularity accompanied to a curvature (Big-Bang) singularity for these models. This is in analogy to ekpyrotic models which have intensively been studied recently.

However, in the anisotropic case, the problem of transition through singularity at weak coupling regime is more complicated and depends on the parameters of the models. This makes some motivation to study such anisotropic models and possibly also some inhomogeneous models within the framework of $\omega = -3/2$ Brans-Dicke cosmology.

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APPENDIX A: CONFORMAL TRANSFORMATIONS

The determinant of the metric $g = \det g_{\mu\nu}$ transforms as

$$\sqrt{-\tilde{g}} = \Omega^4 \sqrt{-g} . \quad (\text{A.1})$$

It is obvious from (II.1) that the following relations for the inverse metrics and the spacetime intervals hold

$$\tilde{g}^{\mu\nu} = \Omega^{-2} g^{\mu\nu} , \quad (\text{A.2})$$

$$d\tilde{s}^2 = \Omega^2 ds^2 . \quad (\text{A.3})$$

The application of (II.1) to the Christoffel connection coefficients gives [33]

$$\tilde{\Gamma}_{\mu\nu}^{\lambda} = \Gamma_{\mu\nu}^{\lambda} + \frac{1}{\Omega} (g_{\mu}^{\lambda} \Omega_{,\nu} + g_{\nu}^{\lambda} \Omega_{,\mu} - g_{\mu\nu} g^{\lambda\kappa} \Omega_{,\kappa}) , \quad (\text{A.4})$$

$$\Gamma_{\mu\nu}^{\lambda} = \tilde{\Gamma}_{\mu\nu}^{\lambda} - \frac{1}{\Omega} (\tilde{g}_{\mu}^{\lambda} \Omega_{,\nu} + \tilde{g}_{\nu}^{\lambda} \Omega_{,\mu} - \tilde{g}_{\mu\nu} \tilde{g}^{\lambda\kappa} \Omega_{,\kappa}) . \quad (\text{A.5})$$

The Ricci tensors and Ricci scalars in the two related frames $g_{\mu\nu}$ and $\tilde{g}_{\mu\nu}$ transform as

$$\tilde{R}_{\mu\nu} = R_{\mu\nu} + \Omega^{-2} [4\Omega_{,\mu} \Omega_{,\nu} - \Omega_{,\sigma} \Omega^{,\sigma} g_{\mu\nu}] - \Omega^{-1} [2\Omega_{;\mu\nu} + \square \Omega g_{\mu\nu}] , \quad (\text{A.6})$$

$$R_{\mu\nu} = \tilde{R}_{\mu\nu} - 3\Omega^{-2} \Omega_{,\rho} \Omega^{,\rho} \tilde{g}_{\mu\nu} + \Omega^{-1} [2\Omega_{;\mu\nu} + \tilde{g}_{\mu\nu} \tilde{\square} \Omega] , \quad (\text{A.7})$$

$$\tilde{R} = \Omega^{-2} \left[R - 6 \frac{\square \Omega}{\Omega} \right] , \quad (\text{A.8})$$

$$R = \Omega^2 \left[\tilde{R} + 6 \frac{\tilde{\square} \Omega}{\Omega} - 12 \tilde{g}^{\mu\nu} \frac{\Omega_{,\mu} \Omega_{,\nu}}{\Omega^2} \right] , \quad (\text{A.9})$$

and the appropriate d'Alembertian operators change under (II.1) as

$$\tilde{\square} \phi = \Omega^{-2} \left(\square \phi + 2g^{\mu\nu} \frac{\Omega_{,\mu}}{\Omega} \phi_{,\nu} \right) , \quad (\text{A.10})$$

$$\square \phi = \Omega^2 \left(\tilde{\square} \phi - 2\tilde{g}^{\mu\nu} \frac{\Omega_{,\mu}}{\Omega} \phi_{,\nu} \right) . \quad (\text{A.11})$$

In these formulas the d'Alembertian $\tilde{\square}$ taken with respect to the metric $\tilde{g}_{\mu\nu}$ is different from \square which is taken with respect to a conformally rescaled metric $g_{\mu\nu}$. Same refers to the covariant derivatives $\tilde{}$ and ; in (A.6)-(A.7).

In order to prove the conformal invariance of the field equations (II.12) it is necessary to know the rule of the conformal transformations for the double covariant derivative of a scalar field, i.e.,

$$\begin{aligned} \tilde{\Phi}_{;\mu\nu} &= \tilde{\Phi}_{,\mu\nu} - \tilde{\Gamma}_{\mu\nu}^{\rho} \tilde{\Phi}_{,\rho} = -\Omega^{-2} \Phi_{;\mu\nu} + \Omega^{-1} \Phi_{;\mu\nu} + 4\Omega^{-3} \Phi \Omega_{,\mu} \Omega_{,\nu} \\ &\quad - 2\Omega^{-2} (\Phi_{,\mu} \Omega_{,\nu} + \Omega_{,\mu} \Phi_{,\nu}) - \Omega^{-3} \Phi g_{\mu\nu} \Omega_{,\rho} \Omega^{,\rho} + \Omega^{-2} g_{\mu\nu} \Phi_{,\rho} \Omega^{,\rho} , \end{aligned} \quad (\text{A.12})$$

and

$$\Phi_{;\mu\nu} = \tilde{\Phi} \Omega_{;\mu\nu} + \Omega \Phi_{;\mu\nu} + \frac{2}{\Omega} \tilde{\Phi} \Omega_{,\mu} \Omega_{,\nu} + 2 \left(\Omega_{,\mu} \tilde{\Phi}_{,\nu} + \tilde{\Phi}_{,\mu} \Omega_{,\nu} \right) - \frac{1}{\Omega} \tilde{\Phi} \tilde{g}_{\mu\nu} \Omega_{,\rho} \Omega^{,\rho} - \frac{1}{\Omega} \tilde{g}_{\mu\nu} \tilde{\Phi}_{,\rho} \Omega^{,\rho} = 0 \quad (\text{A.13})$$

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